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COLLISION OF EIGENVALUES IN LINEAR OSCILLATORY SYSTEMS†

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Autonomous oscillatory systems depending on several parameters are considered. The behaviour of the eigenvalues is considered in the neighbourhood of a multiple point. The method of perturbations is used to show that the interaction of eigenvalues in the neighbourhood of the point in question is described by a family of hyperbolae. The coefficients of the equations of these hyperbolae are calculated using an eigenvector and an associated vector, an eigenvector of the adjoint problem at the multiple point, and the increments of the parameters. The relationships obtained enable an analytical description and geometrical interpretation to be given of two interesting phenomena frequently observed in the literature: the shift of the critical mode of an oscillatory system [1–3], and the destabilization of a system by weak damping [4–8]. Examples considered are the shift of the critical mode of an articulated pipe through which a liquid is flowing, and the destabilization of a non-conservative two-degree-of-freedom Ziegler model.

1. Consider the linear oscillatory system

$$M\ddot{q}+D\dot{q}+Aq=0 \tag{1.1}$$

where M, D and A are real square matrices of order m, representing the inertial, damping and rigidity properties of the system, respectively, and q is an m-vector of generalized coordinates. It is assumed that the elements of M, D and A are smooth functions of the components of the parameter vector $p=(p_1,\ p_2,\ \dots,\ p_n)$ and M is a non-singular matrix.

Seeking a solution of Eq. (1.1) in the form $q=u\exp(\lambda t)$ (t is the time), we obtain a generalized eigenvalue problem

$$[\lambda^2M+\lambda D+A]u=0 \tag{1.2}$$

where λ is an eigenvalue and u is an m-dimensional eigenvector.

Let us assume that the parameter vector $p_0=(p_1^0,\ \dots,\ p_n^0)$ corresponds to an algebraically double eigenvalue λ_0 (a double root of the characteristic equation $\det[\lambda^2M+\lambda D+A]=0$) with an eigenvector u_0 which is unique apart from a factor. This means that the matrix $\lambda^2M+\lambda D+A$ has a deficiency of unity at $\lambda=\lambda_0,\ p=p_0$. In that case the eigenvector u_0 and the associated vector u_1 are determined by the equations

$$Lu_0=0 \tag{1.3}$$

$$Lu_1=-[2\lambda_0M_0+D_0]u_0 \tag{1.4}$$

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$$L = \lambda_0^2 M_0 + \lambda_0 D_0 + A_0, \quad M_0 = M(p_0), \quad D_0 = D(p_0), \quad A_0 = A(p_0) \tag{1.5}$$

where L is a matrix operator. In the complex vector space C^m we define a scalar product (a, b) in the usual way and consider the eigenvalue problem for the operator adjoint to (1.5)

$$L^* \upsilon_0 = 0 \tag{1.6}$$

where $L^* - \overline{L}^T$. Evaluating the scalar product of both sides of (1.4) with υ_0, we obtain

$$([2\lambda_0 M_0 + D_0]u_0, \upsilon_0) = 0 \tag{1.7}$$

since $(Lu_1, \upsilon_0) = (u_1, L^*\upsilon_0) = 0$ because of (1.6).

Let us investigate the behaviour of the eigenvalues λ in the neighbourhood of the point $p = p_0$ in the parameter space R^n. To that end we give the vector p_0 an increment $p = p_0 + \varepsilon k$, where $k = (k_1, k_2, \ldots, k_n)$ is an arbitrary normalized variation vector, $|k| = (k_1^2 + k_2^2 + \ldots + k_n^2)^{1/2} = 1$ and ε is a small parameter, $\varepsilon > 0$. As a result the matrices M, D and A also receive increments

$$M = M_0 + \varepsilon M_1 + \ldots, \quad D = D_0 + \varepsilon D_1 + \ldots, \quad A = A_0 + \varepsilon A_1 + \ldots$$

$$M_1 = \sum_{j=1}^{n} \frac{\partial M}{\partial p_j} k_j, \quad D_1 = \sum_{j=1}^{n} \frac{\partial D}{\partial p_j} k_j, \quad A_1 = \sum_{j=1}^{n} \frac{\partial A}{\partial p_j} k_j \tag{1.8}$$

To solve the spectral perturbation problem (1.2) it is convenient to reduce it, by doubling the dimensions, to the equivalent eigenvalue problem $B\tilde{u} = \lambda \tilde{u}$. This enables us to use the results of [9] concerning perturbation of the spectrum of a non-self-adjoint operator. Using the expansions

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \ldots, \quad u = u_0 + \varepsilon^{1/2} w_1 + \varepsilon w_2 + \ldots \tag{1.9}$$

we obtain an expression for the first correction λ_1 [10, 11]

$$\lambda_1^2 = \sum_{j=1}^{n} f_j k_j \tag{1.10}$$

$$f_j = -([\partial L / \partial p_j]u_0, \upsilon_0)[([2\lambda_0 M_0 + D_0]u_1, \upsilon_0) + (M_0 u_0, \upsilon_0)]^{-1} \tag{1.11}$$

The vectors u_0, u_1 and υ_0 in the last relationship are determined by solving problems (1.3), (1.4) and (1.6), and the derivatives $\partial L / \partial p_j$ are found explicitly

$$\partial L / \partial p_j = \lambda_0^2 \partial M / \partial p_j + \lambda_0 \partial D / \partial p_j + \partial A / \partial p_j \tag{1.12}$$

Note that the coefficients f_j in (1.11) are expressed in terms of values at the point $p = p_0$ and are independent of the variation vector k. Putting

$$a_j = \mathrm{Re}\, f_j, \quad b_j = \mathrm{Im}\, f_j, \quad \Delta p_j = \varepsilon k_j \tag{1.13}$$

and multiplying (1.10) by ε, we obtain

$$\sqrt{\varepsilon}\lambda_1 = \left(\sum_{j=1}^{n} (a_j + ib_j)\Delta p_j \right)^{1/2} \tag{1.14}$$

By (1.9), the right-hand side of (1.14) is the first approximation of the change induced in the double eigenvalue λ_0 (of order $\sqrt{\varepsilon}$) by a parameter variation Δp_j $(j = 1, \ldots, n)$. Note that the expansions (1.9) hold for sufficiently small ε values. Using the normalization $|k|=1$ and the notation Δp_j of (1.13), we obtain a condition for the smallness of ε

$$\varepsilon = \left(\sum_{j=1}^{n} (\Delta p_j)^2 \right)^{1/2} \ll 1 \tag{1.15}$$

To investigate the behaviour of the eigenvalues in the complex λ plane, we introduce notation for the real and imaginary parts of the increment

$$\sqrt{\varepsilon}\lambda_1 = X + iY \tag{1.16}$$

Using (1.16) and squaring (1.14), we obtain

$$X^2 - Y^2 = \sum_{j=1}^{n} a_j \Delta p_j, \quad 2XY = \sum_{j=1}^{n} b_j \Delta p_j \tag{1.17}$$

Eliminating one of the parameters, say Δp_1, from the system of equations (1.17), we have

$$b_1(X^2 - Y^2) - 2a_1 XY = \Delta\varphi = \text{const} \tag{1.18}$$

$$\Delta\varphi = \sum_{j=2}^{n} (b_1 a_j - a_1 b_j)\Delta p_j \tag{1.19}$$

Equation (1.18) for X and Y defines a hyperbola with mutually orthogonal asymptotes $b_1 X = Y(a_1 \pm (a_1^2 + b_1^2)^{1/2})$. Figure 1 shows the asymptotic behaviour of the eigenvalues in the complex λ plane near a double eigenvalue λ_0 with unique eigenvector, for fixed values of Δp_j $(j = 2, \ldots, n)$ and a change in Δp_1. The arrows indicate the "motion" of the eigenvalues when there is a monotonic increase in p_1 in the neighbourhood of p_1^0. The eigenvalues approach one another, merge, and then diverge at right angles to the line of approach. This occurs when $\Delta\varphi = 0$, which by (1.18) means that the hyperbola is degenerate. This is the case, in particular, when all the parameters except p_1 remain unchanged $\Delta p_j = 0$ $(j = 2, \ldots, n)$. For small increments Δp_j $(j = 2, \ldots, n)$, corresponding to a constant $\Delta\varphi \neq 0$ in (1.19), the picture of head-on collision is "spread out". The quadrants that contain the branches of the hyperbola are reversed if the sign of the constant $\Delta\varphi$ is reversed (see Fig. 1).

To construct the hyperbolae (1.18) one uses a solution of the eigenvalue problems (1.3), (1.4) and (1.6) to determine the quantities λ_0, u_0, u_1, v_0; then, using formulae (1.11) and (1.13), one finds the constants a_j, b_j $(j = 1, \ldots, n)$ and constructs the asymptotes. Next,

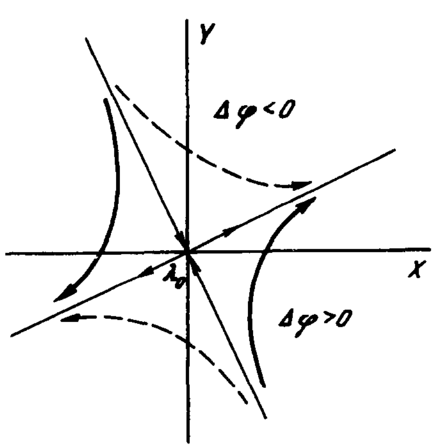

Fig. 1.

determining the increments Δp_j $(j=2, \ldots, n)$ by formulae (1.18) and (1.19), one finally obtains a family of hyperbolae. Note that formulae (1.18) and (1.19) and Fig. 1 provide not only a qualitative picture but also quantitative data for the interaction of the eigenvalues in the neighbourhood of a double root λ_0 with a matrix L of deficiency 1. To investigate the interaction of eigenvalues induced by a change in p_s, with fixed variations Δp_j $(j=1, \ldots, n;$ $j \neq s)$, replace the index 1 in (1.18) and (1.19) by s and extend the summation in (1.19) over all indices $j=1, \ldots, n,$ $j \neq s$.

2. Let us consider the case of two independent parameters p_1 and p_2 in system (1.1) in greater detail. In that case the equation of the hyperbola (1.18), (1.19) becomes

$$b_1(X^2-Y^2)-2a_1XY=\Delta p_2(b_1a_2-b_2a_1) \tag{2.1}$$

Since the parameters p_1 and p_2 are independent by assumption, the vectors (a_1, b_1) and (a_2, b_2) are in general also independent, and so $a_1b_2-a_2b_1 \neq 0$. Hence the hyperbola (2.1) degenerates only when $\Delta p_2=0$, i.e. when only the parameter p_1 is varied. If the sign of Δp_2 is reversed, the quadrants containing the branches of the hyperbola are replaced by the adjacent quadrants. Thus, if $\Delta p_2 \neq 0$ the interaction of the eigenvalues is described by a non-degenerate hyperbola (2.1). This indicates that in a space of two parameters a double complex eigenvalue λ_0 with unique eigenvector is generally an isolated point. The above considerations agree with the results of [12, 13], according to which the singularity of a two-parameter family of real matrices in the case of the general position has a double isolated eigenvalue with a two by two Jordan cell.

Our analysis of the interaction of eigenvalues enables us to describe and explain the phenomenon of critical mode shift, frequently observed in oscillatory systems in parametric investigations. The phenomenon has been observed in studies of the aero-elastic stability (flutter) of aircraft wings, the oscillatory stability of a pipeline through which liquid is flowing, etc. [1–3]. This effect is illustrated qualitatively in Fig. 2, which shows the behaviour of the branches of the eigenvalues when there is a variation of the non-conservative load parameter $p \geqslant 0$. We say that the system S_1 shown in Fig. 2(a) loses stability in the first mode, while the system S_2 obtained from S_1 by a continuous parameter change, as shown in Fig. 2(b), loses stability in the second mode.

As an example, let us consider two-dimensional vibrations of an articulated pipe consisting of three rigid tubular members connected by elastic hinges of stiffness [3], where each member is of length l and mass ml. Thus, the system has two degrees of freedom φ_1, φ_2 and is a Benjamin model of an elastic pipe

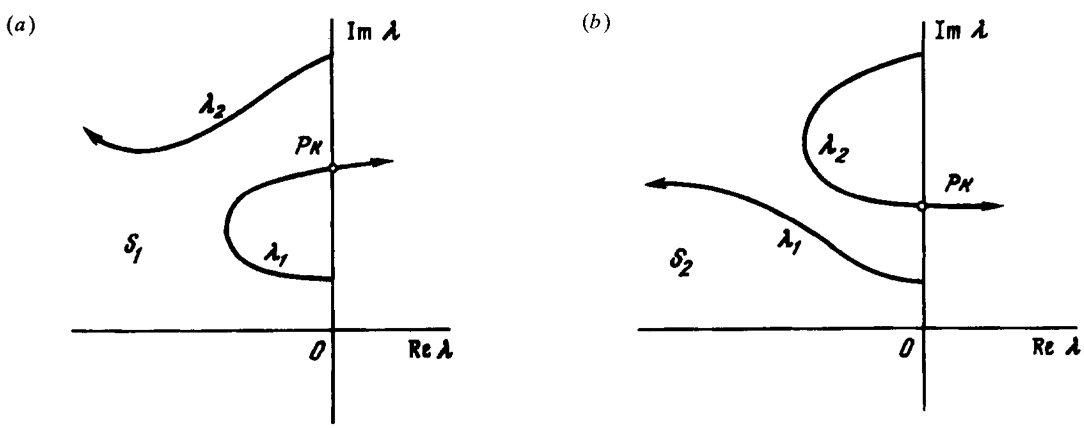

Fig. 2.

through which a liquid is flowing. It is assumed that the liquid is incompressible and is flowing at a constant velocity U relative to the pipe walls. The mass of liquid per unit length of pipe is denoted by m_f; damping is neglected. Then the equations of small vibrations of the whole articulated pipe are [3]

$$\begin{Vmatrix} 4/3 & 1/2 \\ 1/2 & 1/3 \end{Vmatrix} \begin{Vmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{Vmatrix} + \eta p \begin{Vmatrix} 1 & 2 \\ 0 & 1 \end{Vmatrix} \begin{Vmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{Vmatrix} + \begin{Vmatrix} 2-p^2 & p^2-1 \\ -1 & 1 \end{Vmatrix} \begin{Vmatrix} \varphi_1 \\ \varphi_2 \end{Vmatrix} = 0 \tag{2.2}$$

where ρ and η are dimensionless parameters: the flow velocity of the liquid and the relative mass, respectively

$$p = Ul\left(m_f/(cl)^{1/2}\right), \quad p \geqslant 0, \quad \eta = \left(m_f/(m+m_f)\right)^{1/2}, \quad 0<\eta<1 \tag{2.3}$$

The dots in (2.2) denote differentiation with respect to dimensionless time $\tau = t\left(cl(m+m_f)l^3\right)^{1/2}$.

Substituting the expressions $\varphi_j = \xi_j \exp(\lambda\tau)$ $(j=1,\ 2)$ into (2.2) and equating the determinant to zero, we obtain the characteristic equation

$$\lambda^4 + (24/7)\eta p\lambda^3 + (1/7)(108-30p^2+36\eta^2p^2)\lambda^2 + (36/7)\eta p(5-p^2)\lambda + 36/7 = 0 \tag{2.4}$$

The eigenvalues λ depend on the two parameters p and η introduced in (2.3). System (2.2) is stable if $\operatorname{Re}\lambda \leqslant 0$ for all the eigenvalues; otherwise the system is unstable.

A numerical analysis has been carried out [3] of the behaviour of the eigenvalues λ as a function of the velocity parameter p for fixed values of the relative mass parameter η. The critical mode shift phenomenon was observed: at certain η values the first mode was critical, at other values of η the second mode became critical.

To analyse this phenomenon, we first determine the double eigenvalues. If $\lambda = \alpha + i\omega$ is a double eigenvalue, then $\lambda = \alpha + i\omega$ is also a double eigenvalue. Therefore the characteristic equation for the second-order system is

$$(\lambda-\alpha-i\omega)^2(\lambda-\alpha+i\omega)^2 = \lambda^4 - 4\alpha\lambda^3 + 2(3\alpha^2+\omega^2)\lambda^2 - 4\alpha(\alpha^2+\omega^2)\lambda + (\alpha^2+\omega^2)^2 = 0 \tag{2.5}$$

Equating the corresponding terms of Eqs (2.4) and (2.5), we find a unique pair of complex-conjugate double roots $\lambda_0 = \alpha_0 \pm i\omega_0$

$$\begin{aligned} \alpha_0 &= -\left(2\left(1-6/\left(7\sqrt{7}\right)\right)\right)^{1/2} = -1.1628 \\ \omega_0 &= \left(2\left(27/\left(7\sqrt{7}\right)-1\right)\right)^{1/2} = 0.9569 \end{aligned} \tag{2.6}$$

corresponding to parameter values p_0, η_0

$$\begin{aligned} p_0 &= \left(5-4/\sqrt{7}\right)^{1/2} = 1.8677 \\ \eta_0 &= \left(7\left(7\sqrt{7}-6\right)/\left(18\left(5\sqrt{7}-4\right)\right)\right)^{1/2} = 0.7264 \end{aligned} \tag{2.7}$$

Solving Eqs (1.3), (1.4) and (1.6), we evaluate the constants a_j, b_j $(j=1,\ 2)$. Using Eqs (1.14) and (1.16), we deduce from (1.10) and (1.13) that

$$X + iY = \left((a_1+ib_1)\Delta p + (a_2+ib_2)\Delta\eta\right)^{1/2} \tag{2.8}$$

$$a_1 = 6.8832, \quad b_1 = 0.2788; \quad a_2 = 1.3304, \quad b_2 = -6.7855$$

The resulting equation of the hyperbola (2.1) is

$$X^2 - 49.3717XY - Y^2 = 168.8375\Delta\eta \tag{2.9}$$

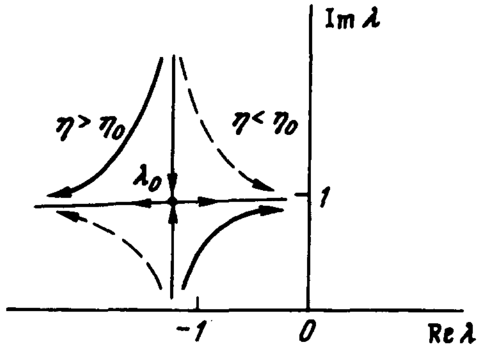

Fig. 3.

Thus, the interaction of the eigenvalues λ in the neighbourhood of the point p_0, η_0 is described by a hyperbola (2.9) with orthogonal asymptotes $X=49.3919Y$ and $X=-0.02024Y$ (see Fig. 3). The arrows indicate the "motion" of λ as p increases. Expressions (2.6)–(2.9) are in complete agreement with the numerical results of [3]. They illustrate the phenomenon of critical mode shift: when $\eta>\eta_0$ the flutter occurs in the first mode; when $\eta>\eta_0$ stability is lost in the second mode.

3. The interaction of the eigenvalues near the double eigenvalue λ_0 also enables us to describe the destabilizing effect of weak damping on a non-conservative system. In this phenomenon, the critical load of a weakly damped system proves to be less than that of the ideal (undamped) system. The effect was observed by Ziegler and has since been investigated by others (see [5–8], etc.).

To describe the destabilization effect, we consider a weakly damped oscillatory system

$$M\ddot{q}+\beta D\dot{q}+A(p)q=0 \tag{3.1}$$

depending on two parameters: the load parameter p and a small parameter $\beta>0$.

The corresponding eigenvalue problem is

$$Lu=0, \quad L=\lambda^2M+\lambda\beta D+A(p) \tag{3.2}$$

Let us first consider an ideal system, i.e. system (3.2) with $\beta=0$. Suppose that at $p=p_0$ a purely imaginary double root $\lambda_0=i\omega_0$ of the characteristic equation $\det[\lambda_0^2M+A(p_0)]=0$ exists, with a unique eigenvector u_0. As we shall show below, the number p_0 is the boundary of the stability domain of the ideal system. By Eqs (1.3), (1.4), (1.6) and (3.2)

$$\left[A(p_0)-\omega_0^2M\right]u_0=0, \quad \left[A^T(p_0)-\omega_0^2M^T\right]\upsilon_0=0, \quad \left[A(p_0)-\omega_0^2M\right]u_1=-2i\omega_0Mu_0 \tag{3.3}$$

Equations (3.3) yield the eigenvector u_0, the associated eigenvector u_1 and the eigenvector υ_0 of the adjoint problem. By (3.2)

$$\partial L/\partial p=\partial A/\partial p, \quad \partial L/\partial\beta=\lambda D \tag{3.4}$$

It follows from Eqs (1.7), (1.10), (1.11) and (3.4) that

$$\varepsilon\lambda_1^2=-\left[([\partial A/\partial p]u_0,\upsilon_0)\Delta p+\lambda_0(Du_0,\upsilon_0)\beta\right]\left(2\lambda_0(Mu_1,\upsilon_0)\right)^{-1} \tag{3.5}$$

Note that by (3.3) the vectors u_0 and υ_0 are real, whereas u_1, which is defined, apart from a term γu_0, $\gamma=\text{const}$, may be chosen to be purely imaginary. Hence we can rewrite (3.5) in the form

$$X+iY=\sqrt{\varepsilon}\lambda_1=(a\Delta p+ib\beta)^{1/2} \tag{3.6}$$

$$a = -([\partial A / \partial p]u_0, \upsilon_0)(2\lambda_0(Mu_1, \upsilon_0))^{-1}, \quad b = -(Du_0, \upsilon_0)(2i(Mu_1, \upsilon_0))^{-1}$$

where a and b are real constants.

Consider relationship (3.6) with $\beta = 0$ and a assumed to be positive. Then as p increases the two eigenvalues λ approach one another along the imaginary axis and merge at $p = p_0$ into a single point $\lambda_0 = i\omega_0$; they then diverge on different sides of a line perpendicular to the imaginary axis (Fig. 4a). The arrows indicate the change in λ as p increases. Thus, p_0 is a critical value for the ideal system: when $p < p_0$ the system is stable, when $p \geqslant p_0$ it is unstable.

If $\Delta p = 0$ and β increases away from zero, the double eigenvalue λ_0 splits along a straight line inclined at an angle of 45° to the imaginary axis. Figure 4(b) illustrates the ray $b > 0$. If $b < 0$, the splitting occurs along a straight line perpendicular to that shown in Fig. 4(b).

If one of the parameters p and β is fixed and the other varied, relationship (3.6) directly implies the equations of hyperbolae

$$X^2 - Y^2 = a\Delta p, \quad 2XY = b\beta \tag{3.7}$$

Figure 5 illustrates the interaction of the eigenvalues when β is fixed and p is allowed to increase. The case of a fixed value of $\Delta p < 0$ and increasing β is shown in Fig. 6. In both figures the solid curves illustrate the case $b > 0$ and the dashed curves illustrate the case $b < 0$. For the construction of the curves shown in Figs 5 and 6 the values of X and Y were found, as follows from (3.7), by solving the equations

$$X^2, Y^2 = \tfrac{1}{2}(\pm a\Delta p + (a^2\Delta p + b^2\beta^2)^{1/2}) \tag{3.8}$$

Relationships (3.6)–(3.8) and the patterns of the interaction of λ as shown in Figs 5 and 6 imply the destabilizing effect of arbitrarily weak damping $\beta > 0$. It is interesting that the same results also imply a destabilizing effect of weak negative damping $\beta < 0$, since by (3.7) reversing the sign of β is equivalent to reversing the sign of the constant b.

4. As an example, let us consider a two-dimensional system representing a double pendulum with viscoelastic hinges [4]. Assume that the torques at the hinges are $c\varphi_1 + d\dot{\varphi}$ and $c(\varphi_2 - \varphi_1) + d(\dot{\varphi}_2 - \dot{\varphi}_1)$, respectively, where c is the stiffness and d the damping coefficient. Point masses $2m$ and m are placed at the middle and end of the pendulum; the arms of the pendulum are both equal to l, and a servo-force P is applied to the system at the free end. Gravitational forces are ignored. The stability of this system was studied by Ziegler [4].

Define dimensionless variables as follows (the magnitude of the servo-force is p, the damping coefficient is β and the time τ)

$$p = Pl / c, \quad \beta = d / (1(cm)^{1/2}), \quad \tau = t(c / (ml^2))^{1/2}$$

The equations of the oscillating system may be written in terms of these variables in the form (3.1) with

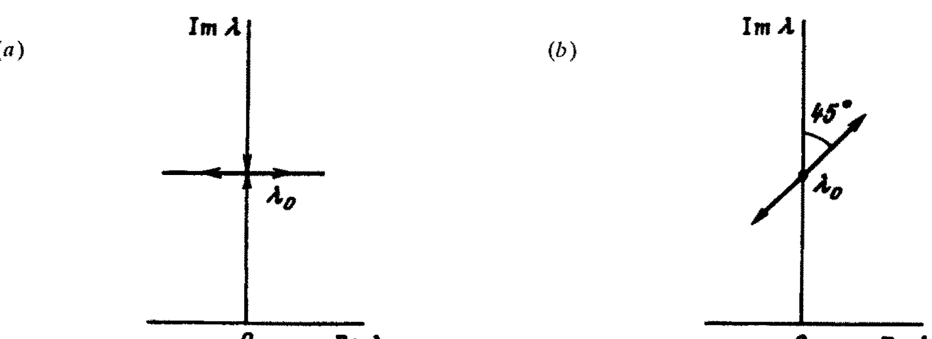

Fig. 4.

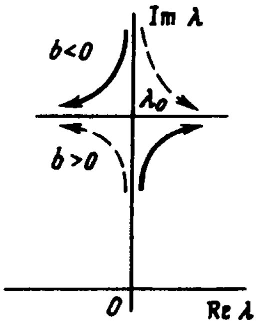

Fig. 5.

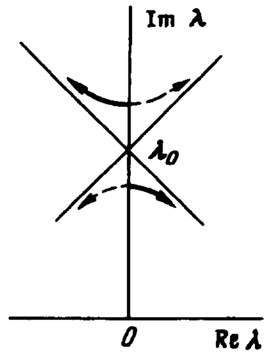

Fig. 6.

$$M = \begin{Vmatrix} 3 & 1 \\ 1 & 1 \end{Vmatrix}, \quad D = \begin{Vmatrix} 2 & -1 \\ -1 & 1 \end{Vmatrix}$$
$$A(p) = \begin{Vmatrix} 2-p & p-1 \\ -1 & 1 \end{Vmatrix}, \quad q = \begin{Vmatrix} \varphi_1 \\ \varphi_2 \end{Vmatrix} \tag{4.1}$$

For a double eigenvalue, we find an isolated point $\lambda_0 = i\omega_0 = i2^{-1/4}$ corresponding to $p_0 = 7/2 - \sqrt{2}$. As p increases away from zero, the two purely imaginary roots λ of the ideal system $(\beta = 0)$ approach one another along the imaginary axis, coalesce at $p = p_0$ into a single point λ_0 and then become complex conjugate points. Thus the ideal system is stable for $0 \leqslant p < p_0$ and unstable for $p \geqslant p_0$.

Now consider the behaviour of the eigenvalues of a system with weak damping $\beta > 0$. We solve Eq. (3.3) with the matrices (4.1) for $p = p_0$, $\lambda - \lambda_0$ and calculate the constants a and b from (3.6); this gives

$$u_0 = \begin{Vmatrix} 3-2\sqrt{2} \\ 1 \end{Vmatrix}, \quad \upsilon_0 = \begin{Vmatrix} 1 \\ \sqrt{2}/2 - 2 \end{Vmatrix}, \quad u_1 = i\begin{Vmatrix} (3-2\sqrt{2})2^{1/4} \\ 0 \end{Vmatrix}$$
$$a = 1/4, \quad b = -(7-2\sqrt{2})/(8 \cdot 2^{1/4}) \tag{4.2}$$

Thus, the asymptotic behaviour of the eigenvalues in Ziegler's problem is described by the hyperbolae (3.7) shown in Figs 5 and 6. Note that the numbers a and b of (4.2) not only provide a qualitative picture but also give quantitative information, provided the expansion parameter (1.15) is small.

5. We have studied the case of a double eigenvalue in detail. A similar study can be made of the more general case of an r-tuple eigenvalue λ_0 with a single linearly independent eigenvector $\bar{u}_0$. Writing the expansions in this case in the form [9]

$$\lambda = \lambda_0 + \varepsilon^{1/r}\lambda_1 + \varepsilon^{2/r}\lambda_2 + \ldots, \quad u = u_0 + \varepsilon^{1/r}w_1 + \varepsilon^{2/r}w_2 + \ldots$$

and reasoning by analogy with the arguments in Section 1, we obtain the asymptotic relations

$$X + iY = \varepsilon^{1/r}\lambda_1 = \left(\sum_{j=1}^{n} (a_j + ib_j)\Delta p_j \right)^{1/r} \tag{5.1}$$
$$a_j = \operatorname{Re} f_j, \quad b_j = \operatorname{Im} f_j$$
$$f_j = -([\partial L/\partial p_j]u_0, \upsilon_0)[([D + 2\lambda_0 M]u_{r-1}, \upsilon_0) + (Mu_{r-2}, \upsilon_0)]^{-1}$$

In these expressions u_0 and υ_0 are the eigenvectors in problems (1.3) and (1.6), while u_1,

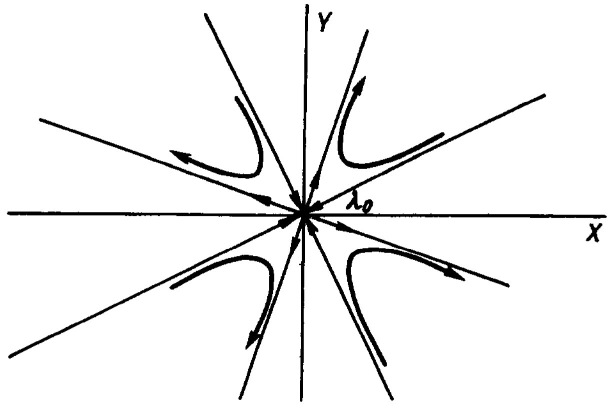

Fig. 7.

$u_2, \ldots, u_{r-1}$ is a sequence of associated eigenvectors satisfying the equations

$$[\lambda_0^2 M + \lambda_0 D + A]u_j + [2\lambda_0 M + D]u_{j-1} + Mu_{j-2} = 0$$
$$j = 1, \ldots, r-1; \quad u_{-1} = 0$$

The interaction of eigenvalues in the complex plane when one parameter p_s is varied and the others are held fixed, Δp_j $(j = 1, \ldots, n;\ j \neq s)$, is illustrated in Fig. 7 for the case $r = 4$. When p_s varies monotonically in the neighbourhood of p_s^0 and $\Delta p_j = 0$ $(j = 1, \ldots, n;\ j \neq s)$, the r eigenvalues converge at angles $2\pi/r$, merge at $p_s = p_s^0$ into a single eigenvalue λ_0, and then diverge along the bisectors of the angles. When the increments Δp_j do not vanish, the pattern of the interaction, generally speaking, is "smeared out", i.e. it is described by non-degenerate hyperbolae.

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REFERENCES

1. BUN'KOV V. G., The computation of optimum flutter characteristics by the gradient method. *Trudy Tsentr. Aero-gidro-dinam. Inst.* 1166, 3–23, 1969.
2. BISHOP R. E. D. and FAWZY I., Free and forced oscillation of a vertical tube containing a flowing fluid. *Phil. Trans. Roy. Soc. London, Ser. A* **284,** 1316, 1–47, 1976.
3. SUGIYAMA Y. and NODA T., Studies on stability of two-degrees-of-freedom articulated pipes conveying fluid. *Bull. ISME.* **24,** 194, 1354–1362, 1981.
4. ZIEGLER H., Die Stabilitätskrieterien der Elastomechanik. *Ing. Arch.* **20,** 1, 49–56, 1952.
5. BOLOTIN V. V., *Non-conservative Problems of the Theory of Elastic Stability.* Fizmatgiz, Moscow, 1961.
6. HERRMANN G., Stability of equilibrium of elastic systems subjected to nonconservative forces. *Appl. Mech. Revs* **20,** 2, 103–108, 1967.
7. PANOVKO Ya. G. and GUBANOVA I. I., *Stability and Vibrations of Elastic Systems.* Nauka, Moscow, 1979.
8. SEIRANYAN A. P., The destabilization paradox in problems of the instability of non-conservative systems. *Uspekhi Mekhaniki* **13,** 2, 89–124, 1990.
9. VISHI M. I. and LYUSTERNIK L. A., Solution of some perturbation problems in the case of matrices and self-adjoint and non-self-adjoint differential equations. *Uspekhi Mat. Nauk* **15,** 3, 3–80, 1960.
10. SEIRANYAN A. P. Sensitivity analysis of eigenvalues and the evolution of instability. *Strojnicky Cas* **42,** 3, 193–208, 1991.
11. SEYRANIAN A. P., Sensitivity analysis of multiple eigenvalues. *Mech. Struct. and Mach.* **21,** 2, 261–284, 1993.
12. GALIN D. M., On real matrices depending on parameters. *Uspekhi Mat. Nauk* **27,** 1, 242–242, 1972.
13. ARNOL'D V. I., *Supplementary Chapters of the Theory of Ordinary Differential Equations.* Nauka, Moscow, 1978.

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